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The Regularity of Div-Curl Systems with Anisotropic Coefficients

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ABSTRACT

This paper investigates the regularity of weak solutions to div-curl systems, with a particular emphasis on anisotropic coefficients characterized by low regularity. The problem of Hölder regularity for the div-curl system with a single anisotropic coefficient, posed by Yin in 2016, has remained open. We fully resolve this long-standing open problem, and our results extend to the case of two anisotropic coefficients as well. We prove that solutions are Hölder-continuous whenever the coefficients are Hölder-continuous. Furthermore, higher Hölder regularity of the coefficients yields a corresponding improvement in the solutions' Hölder regularity. Our results settle Yin's open question and further extend the regularity theory to the case of two anisotropic coefficients. The method developed herein relies on classical tools from partial differential equations—including the Helmholtz decomposition, Campanato space estimates, and uniform ellipticity estimates—and can be further applied to a wider class of linear systems with low-regularity coefficients.

KEYWORDS

div-curl systems, Hölder regularity, anisotropic coefficients

INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected, simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$, and the vector field $u : \Omega \to \mathbb{R}^3$ is a weak solution to the following div-curl system:

$$\begin{cases} \nabla \times (Bu) = f \ \text{in} \ \ \Omega \\ \nabla \cdot (Au) = g \quad \text{in} \ \ \Omega \\ n \times (Bu) = 0 \ \ \text{on} \ \ \partial\Omega \end{cases} \tag{1}$$

where A, B are two matrix-valued functions in $L^{\infty}(\Omega;\mathbb{R}^{3\times3})$. Moreover, there exist constants $0<\lambda_1\leq\lambda_2<\infty$ and $0<\lambda_3\leq\lambda_4<\infty$, such that A, B satisfy:

$$\lambda_1 \mid \xi \mid^2 \leq A\xi\cdot\xi \leq \lambda_2 \mid \xi \mid^2, \quad \lambda_3 \mid \xi \mid^2 \leq B\xi\cdot\xi \leq \lambda_4 \mid \xi \mid^2 \tag{2}$$

The research on the system (1) is closely associated with classical partial differential equations, including the Navier-Stokes equations, Maxwell's equations [1], and linear elasticity theory [2], and it also has important industrial applications in electromagnetic engineering, fluid mechanics and materials science. The uniform ellipticity condition (2) is the core premise for the solvability and regularity analysis of the div-curl system. It ensures the coercivity and boundedness of the coefficient matrices and prevents system degeneracy.The issue of Hölder regularity for the div-curl system with anisotropic coefficients has remained open [3], and most existing studies focus on constant or Lipschitz-continuous isotropic coefficients, with few results involving low-regularity anisotropic coefficients that are more in line with practical engineering scenarios. We commence by investigating prior studies concerning the situation where the coefficient is a numerical value, systematically combing the development of regularity estimates from L^p space to Hölder space, and taking this as a theoretical basis to further explore the regularity of variable anisotropic coefficient cases.

By combining standard div-curl estimates with classical elliptic regularity theory, we present a complete, self-contained solution to the long-standing open problem proposed by Yin [3]. This work further improves the regularity theory of first-order vector systems and provides a unified new approach for addressing related issues in mathematical physics and applied mathematics. The investigation of such systems not only consolidates the theoretical framework of partial differential equations but also furnishes rigorous mathematical foundations for analyzing practical physical problems arising in anisotropic media. By relaxing the smoothness assumptions on coefficients and concentrating on anisotropic structures, our findings help bridge the gap between classical regularity theory and practical engineering applications, rendering the theoretical results more general and adaptable to complex physical models.

Div-Curl Systems with Constant Isotropic Coefficients

In this case, the system takes the following form:

$$\begin{cases} \nabla \times u = f \text{ in } \Omega \\ \nabla \cdot u = g \text{ in } \Omega \\ n \times u = 0 \text{ on } \partial\Omega \end{cases} \tag{3}$$

which is the most basic form of the div-curl system and the foundation for studying all variable- coefficient cases. In 1992, W. von Wahl [4] conducted a pioneering study on the regularity of the solutions to the system (3), and firstly established the classic L^p estimate for the gradient of the weak solution, laying the theoretical foundation for the subsequent regularity analysis of div-curl systems. Subsequently, an L^p estimate for ∇u was derived as follows:

$$\| \nabla u \|_{L^p(\Omega)} \leq C\left(\| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right) \tag{4}$$

where $1 < p < \infty$ and the constant C depends only on Ω and p . This estimate reflects the stability of the constant coefficient div-curl system in the L^p space, and the gradient of the solution can be bounded by the source terms and the solution itself, which is a key result for the subsequent higher-order regularity analysis.

$$\| u \|_{L^{\frac{3p}{3-p}}(\Omega)} \leq C\left(\| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right), 1 < p < 3 \tag{5}$$

$$\| u \|_{C^\alpha(\overline{\Omega})} \leq C\left(\| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right), p > 3 \tag{6}$$

Xiang[5] rigorously showed that the estimate (4) fails to hold when $p =$1, which was achieved through the explicit construction of a carefully designed counterexample. This result clearly defines the applicable range of the classic L^p estimate and indicates that the div-curl system has different regularity properties in the low-exponent L^1 space, providing a critical reference for the study of non-reflexive space regularity. Later, in 2005, Bougain and Brezis [6] made a significant contribution by establishing the following crucial $L^{\frac{3}{2}}$ estimates for the div-curl system, which precisely correspond to the specific case of (5) where $p = 1$ and $g = 0$:

$$\| u \|_{L^{3/2}(R^3)} \leq C\left(\| f \|_{L^1(R^3)} \right) \tag{7}$$

Hong-Ming Yin [3] improved the estimate (6) in 2016 and obtained the following sharp Hölder regularity for the solution to the constant coefficient div-curl system:

$$\| u \|_{c^{\alpha}(\overline{\Omega})} \leq C \left(\| f \|_{L^{2,\tau}(\Omega)} + \| g \|_{L^{2,\tau}(\Omega)} \right) \tag{8}$$

where $\alpha = \frac{\tau - 1}{2}$ for some $\tau \in (0,3)$ and C depends only on Ω and τ .

It should be noted that the methods proposed by Shen and von Wahl typically require coefficients to exhibit high regularity, such as Lipschitz continuity or even smooth differentiability. However, Hölder-continuous coefficients only satisfy weaker regularity conditions (i.e., the existence of constants $C > 0$ and $\alpha \in (0,1]$ such that $\mid f(x) - f(y) \mid \leq C \mid x - y \mid^{\alpha}$), which may render certain critical estimates in their methods invalid. For example, in the finite element analysis of elliptic equations, their approach relies on coefficient smoothness to establish stability and convergence estimates, whereas Hölder-continuous coefficients may fail to meet the prerequisites for these estimates.

HÖLDER REGULARITY OF DIV-CURL SYSTEM WITH PARTIALLY ANISOTROPIC VARIABLE COEFFICIENTS

The study of partially anisotropic variable coefficient div-curl systems relies on two core theoretical tools in vector-field analysis and elliptic regularity theory: the Helmholtz decomposition and Campanato space theory. The Helmholtz decomposition serves as the fundamental bridge for transforming the first-order vector div-curl system into a second-order scalar elliptic equation, which allows us to apply the well-established elliptic estimate results to the study of div-curl systems. Meanwhile, the Campanato space theory provides a precise and flexible way to characterize the Hölder continuity of functions from the perspective of integral average oscillation, which is essential for proving the Hölder regularity of weak solutions for low-regularity coefficient systems. In this section, we study the system with partially anisotropic variable coefficients where only the matrix-valued coefficient $A(x)$ is a variable anisotropic function and $B(x)$ is a constant matrix, as shown below:

$$\{ \nabla \times u = f \quad \text{in} \quad \Omega$$

$$\nabla \cdot (A(x)u) = g \quad \text{in} \quad \Omega \tag{9}$$

$$n \times u = 0 \quad \text{on} \quad \partial\Omega$$

In 2014, Shen [7, 8] deduced the subsequent estimation of the solution in L^p to the system (9) as presented below:

$$\| \nabla u \|_{L^p(\Omega)} \leq C \left(\| \nabla \times u \|_{L^p(\Omega)} + \| \nabla \cdot (Au) \|_{L^p(\Omega)} \right) \tag{10}$$

where $1 < p < \infty$, and $A(x)$ is Lipschitz-continuous.

This estimation extends the $W^{1,p}$ regularity of the constant coefficient system to the Lipschitz-continuous anisotropic coefficient case, but it has strict requirements on the smoothness of the coefficient $A(x)$. In the case where $A(x)$ is only Hölder-continuous (i.e., $W^{1,p} \in C^{0,\alpha}(\Omega)$), the $W^{1,p}$ estimate of the solutions to (9) is still unclear, and the classic difference methods and integral estimate techniques for Lipschitz coefficients can no longer be directly applied due to the lack of first-order differentiability. Furthermore, Yin [3] claimed in his research that the Hölder regularity of the weak solution to the div-curl system (9) with Hölder-continuous anisotropic coefficients is still an open problem, which is the core research starting point of this paper. In this paper, we study the regularity of solutions to the fully anisotropic system (1) under the assumption that both inhomogeneous anisotropic coefficients $A(x)$ and $B(x)$ are Hölder-continuous. We first establish complete regularity results for the partially anisotropic coefficient case (9) in this section by improving the classic Helmholtz decomposition and combining it with Campanato space elliptic estimates, followed by a natural extension of these results to the fully anisotropic system in the subsequent section through a reasonable variable transformation strategy.

Before presenting the main results for partially anisotropic coefficients, we first recall the key tools that will be used throughout the paper. The Helmholtz decomposition serves as the foundation for transforming the original div-curl system into a second-order elliptic equation, which allows us to use well-established elliptic estimates. Meanwhile, the Campanato space theory provides a precise way to characterize the Hölder continuity of solutions, which is essential for proving the main regularity theorems.

Lemma 2.1 (Helmholtz decomposition). Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected, simply connected open set with a $C^{1,1}$ -smooth boundary ∂Ω, and let n denote the unit outward normal vector to ∂Ω. If $u \in L^2(\Omega; \mathbb{R}^3)$, then there exists a scalar function $q \in H_0^1(\Omega)$ and a vector function $\phi \in H^1(\Omega)$ s.t.

$$u = \nabla q + \mathrm{curl} \quad \phi \quad \text{in} \quad \Omega \tag{11}$$

where ϕ satisfies that

$$\begin{cases}\text{div} \quad \phi = 0 \text{ in } \quad \Omega \\ \phi \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{12}$$

and

$$\| \phi \quad \|_{H^1(\Omega)} \leq C \| u \quad \|_{L^2(\Omega)} \tag{13}$$

where $C > 0$ depends only on Ω .

This lemma plays a fundamental role in the analysis of div-curl systems, as it uniquely decomposes any square-integrable vector field into a gradient part of a scalar function and a divergence-free curl part, which is the key to decoupling the complex vector relationship between the curl and div terms of the solution u . Such a decomposition not only transforms the first-order div-curl system into a solvable second-order elliptic equation for the scalar function ϕ , but also preserves the regularity of the original vector field u in the decomposition process, which is the starting point of all regularity estimates in this paper. The boundedness estimate (13) is particularly important, as it ensures that the norm of the decomposed parts can be controlled by the norm of the original vector field, avoiding the loss of regularity in the decomposition and laying the foundation for the subsequent derivation of all solution estimates.This estimate is a classical result for the Helmholtz decomposition in bounded, simply connected C1,1 domains in R3. Girault and Raviart[9] provided a comprehensive proof and detailed discussion of it in 1986.

Furthermore,the decomposition u is not unique, and the boundary conditions on q and ϕ need to be carefully chosen to satisfy the original boundary condition:

It is known that the condition $\mathbf{n} \times \mathbf{u} = 0 \quad \text{holds on} \quad \partial\Omega$, and u can be decomposed as (11).

Substituting the decomposition into the boundary conditions yields: $n \times (\nabla q + \nabla \times \phi) = 0$

Using the vector identity theorem, we expand as follows:

1. The first term, $\mathbf{n} \times \nabla q = \nabla q \times \mathbf{n}$ (applying the commutative law).
2. The second term, $n \times (\nabla \times \phi) = (n \cdot \nabla)\phi - \nabla(n \cdot \phi)$.

Substituting this into the original equation gives: $\nabla q \times n + (n \cdot \nabla)\phi - \nabla(n \cdot \phi) = 0$.

To satisfy this equation, we adopt the following boundary conditions:

1. Let $\nabla q \times n = 0$, implying that the gradient of the scalar potential q has no tangent component along the boundary normal.
2. Let $(n \cdot \nabla)\phi - \nabla(n \cdot \phi) = 0$; using the vector identity theorem, this simplifies to:

$n \times (\nabla \times \phi) = 0$.

Thus, the final boundary conditions are:
$\{\nabla q \times n = 0 \quad \text{on} \quad \partial\Omega$

$n \times (\nabla \times \phi) = 0 \quad \text{on} \quad \partial\Omega$

This boundary condition will be used multiple times in the subsequent sections; therefore, a brief explanation is provided here.

Theorem 2.2. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$, $A(x) \in L^\infty(\Omega)$,and $f, g \in L^2(\Omega)$, then the system (9) admits at most one solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u \|_{L^2(\Omega)} \leq C \left(\| f \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right) \tag{14}$$

where the constant C depends only on Ω and $\| A \|_{L^\infty(\Omega)}$.

Theorem 2.2 provides the basic L^2 estimate for weak solutions to the partially anisotropic div-curl system, which is the first step in the regularity analysis of the variable coefficient system. It shows that the weak solution to the system (9) is unique under the uniform ellipticity condition of the coefficient matrix $A(x)$, and $\| u \|_{L^2(\Omega)}$ can be bounded by the given source terms f and g , with the estimate constant only depending on the domain Ω and the ellipticity constants λ, Λ of $A(x)$. This estimate serves as a preliminary result for the higher-order regularity analysis in subsequent theorems, and its proof fully combines the Helmholtz decomposition of the solution u and the uniform ellipticity of $A(x)$, which is a typical method for studying the solvability of linear partial differential equations. More importantly, the uniqueness conclusion in Theorem 2.2 ensures the well-posedness of the partially anisotropic div-curl system, which is a necessary prerequisite for all subsequent regularity analysis and practical applications.

Proof. Apply the Helmholtz decomposition to solution u , then u can be written as the form of equation (11). Through the application of integration by parts, it is obtained that,

$$\begin{aligned}&\int_\Omega \text{curl } (\nabla q)\cdot\phi-\nabla q\cdot \text{curl } \ \phi\, dx\\ &=\int_\Omega q \text{ div } (\text{curl } \ \phi)\, dx-\int_{\partial\Omega} q \text{ curl } \ \phi\cdot n\, ds=0, \quad \forall\phi\in C^\infty(\Omega,R^3)\end{aligned} \tag{15}$$

It means that $\nabla q\times n=0$ on $\partial\Omega$, and $n\times\nabla\times\phi=n\times(u-\nabla q)=0$ on $\partial\Omega$.

Then,we can get the following div-curl form equation for ϕ .

$$\begin{cases}\nabla\times\nabla\times\phi=f \text{ in } \ \Omega\\ \nabla\cdot\nabla\times\phi=0 \ \text{ in } \ \Omega\\ n\times\nabla\times\phi=0 \ \text{ on } \ \partial\Omega\end{cases} \tag{16}$$

By applying inequality (4) for $\nabla\times\phi$ with $p=2$,we have:

$$\| \nabla(\nabla\times\phi) \|_{L^2(\Omega)} \le C\left(\| f \|_{L^2(\Omega)}\right) \tag{17}$$

At the same time ∇q satisfies that.

$$\begin{cases}\nabla\cdot(A\nabla q)=-\nabla\cdot(A\nabla\times\phi)+g \text{ in } \ \Omega\\ q=0 \qquad\qquad\qquad\qquad\qquad\quad \text{ on } \ \partial\Omega\end{cases} \tag{18}$$

And the weak formulation of (18) can be written as follows.

$$\int_\Omega -(A\nabla q)\cdot\nabla q\, dx=\int_\Omega (A\nabla\times\phi)\cdot\nabla q+gq\, dx \tag{19}$$

The subsequent estimate for ∇q can be derived through the utilization of the Cauchy inequality and the Poincaré inequality by applying the uniform elliptic property of the coefficient A .

$$\lambda_1 \| \nabla q \|^2_{L^2(\Omega)} \le C\left(\lambda_2 \| \nabla\times\phi \|_{L^2(\Omega)} \| \nabla q \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \| \nabla q \|_{L^2(\Omega)}\right) \tag{20}$$

Finally, the estimates (17) and (20) lead to the following estimate for u as indicated in (14).

Theorem 2.2 provides the basic L^2 estimate for weak solutions to the partially anisotropic div-curl system. It shows that the solution can be bounded by the given source terms under the uniform ellipticity condition on the coefficient matrix $A(x)$. This estimate serves as a preliminary result for the higher-order regularity analysis in subsequent theorems.

Remark 2.3. By expressing the (18) in Einstein notation in the following manner.

$$\nabla \cdot (A\nabla q) = \big(\partial_i A_{ij}\big)(\nabla \times \phi)_j - \big(A_{ij}\big)\big(\partial_i (\nabla \times \phi)_j\big) + g \tag{21}$$

By applying Sobolev embedding, $(\nabla \times \phi)_j \in L^6(\Omega)$, together with $\partial_i A_{ij} \in L^3(\Omega)$, we can obtain that the right-hand side of the equation belongs to $L^2(\Omega)$, and hence $u \in H^1(\Omega)$.

Theorem 2.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$, $A(x) \in C^\alpha(\overline{\Omega})$ for $\alpha \in (0,1)$, and $f, g \in L^p(\Omega)$ for $p \geq \frac{3}{1-\alpha}$, then the system (9) admits at most one Hölder-continuous solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u \|_{C^\alpha(\bar{\Omega})} \leq C\left(\| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right) \tag{22}$$

where the constant C depends only on Ω and $\| A \|_{C^\alpha(\overline{\Omega})}$.

Theorem 2.4 gives the first Hölder regularity result for the partially anisotropic system. It indicates that the solution inherits the Hölder continuity from the coefficient $A(x)$, which is exactly the open problem proposed by Yin. This theorem confirms that the Hölder regularity holds under minimal smoothness assumptions on the coefficient.

Remark 2.5. Theorem 2.6 (formulated subsequently) is a more generalized version of Theorem 2.4 , and hence we provide the proof of Theorem 2.6 only.

Theorem 2.6. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$, $A(x) \in C^\alpha(\overline{\Omega})$, $\alpha = \frac{\tau-1}{2}$ and $f, g \in L^{2,\tau}(\Omega)$ for $\tau \in (1,3)$, then, the system (9) admits at most one Hölder-continuous solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u \|_{C^{\alpha}(\bar{\Omega})} \leq C\left(\| f \|_{L^{2,\tau}(\Omega)} + \| g \|_{L^{2,\tau}(\Omega)} \right) \tag{23}$$

where the constant C depends only on Ω and $\| A \|_{C^{\alpha}(\overline{\Omega})}$.

The proof for Theorem 2.6 relies on elliptic estimates in Campanato spaces [10]. We therefore present a short introduction of the related results before proceeding to the formal proof. We denote the Campanato spaces with $L^{p,\lambda}(\Omega)$, and its norm can be given by:

$$\| u \|_{L^{2,\lambda}(\Omega)} = \| u \|_{L^{2}(\Omega)} + \left\{ \sup_{x\in\Omega, 0<r< \text{ diam } (\Omega)} r^{-\lambda} \int_{\Omega(x,r)} \left| u(z) - \bar{u}_{x,r} \right|^{2} dz \right\}^{\frac{1}{2}} \tag{24}$$

where $\Omega(x,r) = \Omega \cap \{ y \in \mathbb{R}^3 : \mid y - x \mid < r \}$ is the average of \$ and $\bar{u}_{x,r} = \frac{1}{\mid \Omega(x,r) \mid} \int_{\Omega(x,r)} u(z)\, dz$.

If $\lambda \in (3,5)$,then $L^{2,\lambda}(\Omega) \hookrightarrow C^{0,\frac{\lambda-3}{2}}(\overline{\Omega})$ and the embedding $L^{p}(\Omega) \to L^{\frac{2,3p-2}{p}}(\Omega)$ is continuous.

If $\lambda < 3, \quad u \in L^2(\Omega)$ and $\nabla u \in L^{2,\lambda}(\Omega)$, then $L^{p,3}(\Omega) \hookrightarrow C^{0,1-3/p}(\overline{\Omega})$ and $the u \in L^{2,2+\lambda}(\Omega)$ embedding is continuous.

The Campanato space is a powerful generalization of both the Lebesgue space and the Hölder space, and it characterizes the regularity of functions by measuring the average oscillation of the function on all balls contained in the domain, which is more flexible than the direct pointwise difference estimate for the study of low-regularity function spaces. In the proof of Theorem 2.6, the application of Campanato space theory avoids the complex direct estimation of the pointwise difference of the solution u , and transforms the Hölder regularity problem of the weak solution into a Campanato space embedding problem, which greatly simplifies the proof process and makes the conclusion more general and applicable to low-regularity coefficient cases.

Proof. Let $u \in L^2(\Omega)$ be a weak solution to system (9). By the Helmholtz Decomposition,

u can be written as the form of equation (11), moreover, $\nabla \times \phi$ satisfies the system (16)

where $f \in L^{2,\tau}(\Omega)$. Apply (8) to $\nabla \times \phi$, we can obtain that

$$\| \nabla \times \phi \|_{C^{\alpha}(\bar{\Omega})} \leq C\left(\| f \|_{L^{2,\tau}(\Omega)} \right) \tag{25}$$

at the same time, ∇q meets the condition (18) where $A, \nabla \times \phi \in C^{\alpha}(\overline{\Omega})$ and $g \in L^{2,\tau}(\Omega)$.

Then the classic $L^{2,\tau}$ estimate leads that

$$\| \nabla q \|_{L^{2,\tau+2}(\Omega)} \leq C\left(\| A\nabla \times \phi \|_{L^{2,\tau+2}(\Omega)} + \| g \|_{L^{2,\tau}(\Omega)} + \| q \|_{H^1(\Omega)} \right) \tag{26}$$

From the Theorem(2.2), we can obtain $\| q \|_{H^1(\Omega)} \leq C\left(\| f \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right)$, and in consideration of $L^{2,t+2}(\Omega) \cong C^{0,\frac{t-1}{2}}(\overline{\Omega})$, we can elicit that

$$\| u \|_{C^{\alpha}(\bar{\Omega})} \leq C\left(\| f \|_{L^{2,\tau}(\Omega)} + \| g \|_{L^{2,\tau}(\Omega)} \right) \tag{27}$$

Theorem 2.7. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{k+1,1}$ -smooth connected boundary $\partial\Omega$, $A(x) \in C^{k,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$. Suppose that $f, g \in C^{k-1,\alpha}(\overline{\Omega})$ for $k \in [1,\infty)$, then the system (9) admits at most one solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u \|_{C^{k,\alpha}(\bar{\Omega})} \leq C\left(\| f \|_{C^{k-1,\alpha}(\overline{\Omega})} + \| g \|_{C^{k-1,\alpha}(\overline{\Omega})} \right) \tag{28}$$

where the constant C depends only on Ω and $\| A \|_{C^{k,\alpha}(\overline{\Omega})}$.

Proof. The proof is similar to that of Theorem 2.6, $\nabla \times \phi$ satisfies the system (16), where $f \in C^{k-1,\alpha}(\overline{\Omega})$, from the higher-order Hölder regularity of the div-curl system where the coefficients are simply numbers, we can derive

$$\| \nabla \times \phi \|_{C^{k,\alpha}(\bar{\Omega})} \leq C\left(\| f \|_{C^{k-1,\alpha}(\overline{\Omega})} \right) \tag{29}$$

And ∇q meets the condition (18) where $A, \nabla \times \phi \in C^{k,\alpha}(\overline{\Omega})$ and $g \in C^{k-1,\alpha}(\overline{\Omega})$.

By the classic Schauder estimate,we then have that

$$\| \nabla q \|_{C^{k+1,\alpha}(\bar{\Omega})} \leq C \left(\| A\nabla \times \phi \|_{C^{k,\alpha}(\overline{\Omega})} + \| g \|_{C^{k-1,\alpha}(\overline{\Omega})} + \| q \|_{H^1(\Omega)} \right) \tag{30}$$

And finally

$$\| u \|_{C^{k,\alpha}(\bar{\Omega})} \leq C \left(\| f \|_{C^{k-1,\alpha}(\overline{\Omega})} + \| g \|_{C^{k-1,\alpha}(\overline{\Omega})} \right) \tag{31}$$

Theorem 2.7 further improves the regularity to higher-order Hölder spaces. It shows that if the coefficient has higher smoothness, the solution will automatically gain the same level of smoothness. This result reveals a strong regularity inheritance property of the anisotropic div-curl systems.

DIV-CURL SYSTEMS WITH FULLY ANISOTROPIC IN HOMOGENEOUS COEFFICIENTS

In this section, we extend our regularity results from the partially anisotropic case to the fully anisotropic case, where both coefficient matrices $A(x)$ and $B(x)$ are non-constant anisotropic functions satisfying the uniform ellipticity condition (2), which is the most general form of the div-curl system studied in this paper and is more consistent with the behavior of anisotropic media in practical engineering. The core research method adopted in this section is to perform a simple and reasonable variable substitution to reduce the fully anisotropic system with two variable coefficients into the partially anisotropic one studied in Section 2, so that all the previous regularity estimates and conclusions for the partially anisotropic case can be directly applied to the fully anisotropic system. This reduction method is concise and effective, and it fully preserves the regularity characteristics of the original system in the transformation process, which can also be adapted to other first-order vector systems with multiple anisotropic coefficients. Based on this method, we conduct a study on the Hölder regularity of the following fully anisotropic div-curl system:

$$\begin{cases} \nabla \times (B(x)u) = f \text{ in } \Omega \\ \nabla \cdot (A(x)u) = g \text{ in } \Omega \\ n \times (B(x)u) = 0 \text{ on } \partial\Omega \end{cases} \tag{32}$$

where $A(x)$ and $B(x)$ are both Hölder-continuous anisotropic matrix-valued functions satisfying the uniform ellipticity condition (2).

Corollary 3.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$, $A(x), B(x) \in L^\infty(\Omega)$,and $f, g \in L^2(\Omega)$. Assume that $u \in L^2(\Omega)$ is a weak solution to system (32) , $u(x)$ satisfies the following estimate:

$$\| u \|_{L^2(\Omega)} \leq C\left(\| f \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right) \tag{33}$$

where the constant C depends only on Ω , $\| A \|_{L^\infty(\Omega)}$ and $\| B \|_{L^\infty(\Omega)}$.

Corollary 3.1 extends the basic L^2 estimate to the fully anisotropic div-curl system (32), which is the first regularity result for the most general case of the div-curl system. By a simple variable substitution $\tilde{u} = Bu$,we reduce the two-coefficient fully anisotropic system to the one-coefficient partially anisotropic system studied in Section 2, which means all the previous $L^2, W^{1,p}$ and Hölder regularity estimates remain valid for the more general fully anisotropic system. The estimate constant in (33) not only depends on the domain Ω and the ellipticity constant of $A(x)$, but also on the ellipticity constant of $B(x)$, which is a natural generalization of the partially anisotropic case. This result confirms the stability of the L^2 regularity of the div-curl system with double anisotropic coefficients and the well-posedness of the fully anisotropic system under the uniform ellipticity condition, laying the foundation for the subsequent Hölder regularity analysis of the fully anisotropic case.

Proof. Let $\tilde{u} = Bu$,and (32) can be transformed into the following system.

$$\begin{cases} \nabla \times \tilde{u} = f & \text{in } \ \Omega \\ \nabla \cdot (AB^{-1}\tilde{u}) = g & \text{in } \ \Omega \\ n \times \tilde{u} = 0 & \text{on } \ \partial\Omega \end{cases} \tag{34}$$

where (34) belongs to the class of systems in Section 2. By applying Theorem (10),(17),(20), and $AB^{-1} \in L^\infty(\Omega)$, then we can get that.

$$\| u \|_{L^2(\Omega)} = \| B^{-1}(Bu) \|_{L^2(\Omega)} \leq C \| Bu \|_{L^2(\Omega)} \leq C\left(\| f \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right) \tag{35}$$

Remark 3.2. When considering $B \in W^{1,\infty}(\Omega)$ and $A \in W^{1,3}(\Omega)$, the solution $u(x)$ also satisfies the following estimate:

$$\| \nabla u \|_{L^2(\Omega)} = \| \nabla(B^{-1}Bu) \|_{L^2(\Omega)} \leq C \| \nabla(Bu) \|_{L^2(\Omega)} \leq C\left(\| f \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \right) \tag{36}$$

Corollary 3.3. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$,the coefficients $A(x), B(x) \in C^{\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$, and $f, g \in L^p(\Omega)$ for $p \geq \frac{3}{1-\alpha}$, then the system (32) admits at most one Hölder-continuous solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u \|_{C^{\alpha}(\bar{\Omega})} \leq C\left(\| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right) \tag{37}$$

where the constant C depends only on Ω and $\| A \|_{C^{\alpha}(\overline{\Omega})}$, $\| B \|_{C^{\alpha}(\overline{\Omega})}$.

Corollary 3.3 establishes the Hölder continuity for the fully anisotropic div-curl system with double Hölder-continuous anisotropic coefficients, which is the core result of Section 3 and the key conclusion of this paper. This corollary generalizes the Hölder regularity result from the single anisotropic coefficient case to the two anisotropic coefficients case, and it confirms that the Hölder regularity of the weak solution is stable even if both $A(x)$ and $B(x)$ are variable and anisotropic. More importantly, this result completely solves the open problem proposed by Yin [3] and its natural generalization, enriching the complete regularity theory of the anisotropic div-curl system and providing a rigorous theoretical basis for the study of div-curl systems in complex anisotropic media.

Proof. From Theorem 2.4, and since $AB^{-1} \in C^{\alpha}(\overline{\Omega})$, we can obtain that.

$$\| u \|_{C^{\alpha}(\overline{\Omega})} = \| B^{-1}(Bu) \|_{C^{\alpha}(\overline{\Omega})} \leq C \| Bu \|_{C^{\alpha}(\overline{\Omega})} \leq C\left(\| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right) \tag{38}$$

Corollary 3.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$,the coefficients $A(x), B(x) \in C^{\alpha}(\overline{\Omega})$ for $\alpha = \frac{\tau-1}{2}$, and $f, g \in L^{2,\tau}(\Omega)$ for $\tau \in$

$(1,3)$, then the system (32) admits at most one Hölder-continuous solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u(x) \|_{C^{\alpha}(\overline{\Omega})} \leq C\left(\| f \|_{L_{\lambda}^{2,t}(\Omega)} + \| g \|_{L_{\lambda}^{2,t}(\Omega)}\right) \tag{39}$$

Proof. From Theorem 2.6, and $AB^{-1} \in C^{\alpha}(\overline{\Omega})$.

$$\| u \|_{C^{\alpha}(\overline{\Omega})} = \| B^{-1}(Bu) \|_{C^{\alpha}(\overline{\Omega})} \leq C \| Bu \|_{C^{\alpha}(\overline{\Omega})} \leq C\left(\| f \|_{L^{2,\mu}(\Omega)} + \| g \|_{L^{2,\mu}(\Omega)}\right) \tag{40}$$

Corollary 3.5. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{k+1,1}$ -smooth connected boundary $\partial\Omega$,and $A(x) \in C^{k,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$,and $f, g \in C^{k-1,\alpha}(\overline{\Omega})$, for $k \geq 1$, then the system (32) admits at most one solution. Furthermore, the solution $u(x)$ satisfies the following estimate:

$$\| u \|_{C^{k,\alpha}(\overline{\Omega})} \leq C\left(\| f \|_{C^{k-1,\alpha}(\overline{\Omega})} + \| g \|_{C^{k-1,\alpha}(\overline{\Omega})}\right) \tag{41}$$

Proof. By applying Theorem 2.7, and $(A(x)B(x))^{-1} \in C^{k,\alpha}(\overline{\Omega})$, we obtain that

$$\| u \|_{C^{k,\alpha}(\overline{\Omega})} = \| B^{-1}(Bu) \|_{C^{k,\alpha}(\overline{\Omega})} \leq C \| Bu \|_{C^{k,\alpha}(\overline{\Omega})} \leq C\left(\| f \|_{C^{k-1,\alpha}(\overline{\Omega})} + \| g \|_{C^{k-1,\alpha}(\overline{\Omega})}\right) \tag{42}$$

APPLICATION TO THE MAXWELL'S SYSTEMS

The time-harmonic Maxwell system is one of the most important physical models described by the div-curl system, and its regularity theory has attracted extensive attention from mathematicians and physicists over the past decades. Many important physical phenomena, such as electromagnetic wave propagation, electromagnetic resonance and energy transfer in anisotropic media, are closely related to the smoothness of the electric and magnetic field solutions to the Maxwell system. The study of the Hölder regularity of the Maxwell system is of great significance for the analysis of the propagation characteristics of electromagnetic waves in anisotropic materials such as optical fibers and magnetic crystals. By applying our main regularity theorems of the anisotropic div-curl system to the time-harmonic Maxwell system, we obtain new sharp Hölder regularity results that improve and extend some existing conclusions in the literature, and these results make the regularity theory of the Maxwell system more applicable to practical engineering scenarios.

Let E represent the electric field, and H represent the magnetic field, suppose $(E, H) \in H(\mathrm{curl}, \Omega) := \{F \in L^2(\Omega) : \mathrm{curl} \quad F \in L^2(\Omega)\}$ is a weak solution to the following time-harmonic Maxwell's system in the frequency domain:

$$\{\mathrm{curl}\ H = i\omega\varepsilon E + J_e \qquad \text{in} \ \ \Omega$$
$$\mathrm{curl}\ E = -i\omega\mu E + J_m \qquad \text{in} \ \ \Omega \tag{43}$$
$$\mathbf{n} \times E = \mathbf{n} \times G \qquad \text{on} \ \ \partial\Omega$$

Where $\varepsilon(x), \mu(x)$ are matrix-valued functions in $L^\infty(\mathbb{R}^3)$ representing the electric permittivity and magnetic permeability of the anisotropic medium respectively; the terms $J_m, J_e \in L^2(\Omega)$ represent current sources.Maxwell system (43) can be written in terms of the magnetic field H alone by eliminating the electric field E , as follows:

$$\{\nabla \times (\varepsilon^{-1} \nabla \times H) - \omega^2 \mu u = i\omega J_m + \nabla \times (\varepsilon^{-1} J_e) \qquad \text{in} \ \ \Omega$$
$$\nabla \cdot (\nabla \times H) = \nabla \cdot J_e \qquad \text{in} \ \ \Omega \tag{44}$$
$$\mathbf{n} \times (\varepsilon^{-1} \nabla \times H) = \mathbf{n} \times (i\omega G + \varepsilon^{-1} J_e) \qquad \text{on} \ \ \partial\Omega$$

Owing to the symmetric structure of the system (44), the electric field E also adheres to a similar single-field div-curl type system after elimination. Therefore, we conduct a study on the following more general scalar div-curl type system that unifies the electric and magnetic field equations:

$$\{\nabla \times (A(x) \nabla \times u) + B(x) u = f + \nabla \times g \qquad \text{in} \ \ \Omega$$
$$\nabla \cdot (\nabla \times u) = h \qquad \text{in} \ \ \Omega \tag{45}$$
$$\mathbf{n} \times \nabla \times u = \mathbf{n} \times G \qquad \text{on} \ \ \partial\Omega$$

The research on equation (45) has a long-standing history in the field of mathematical physics. The works in [11–15] investigated the Hölder regularity of the solution through the utilization of $W^{1,p}$ estimates and classical elliptic regularity theory, where the coefficient matrices $A(x)$ and $B(x)$ are Lipschitz-continuous and with certain additional structural constraints $1 < p < \infty$ imposed on the coefficients. Subsequently, Yin [16] derived analogous Hölder regularity results for the Maxwell system under the assumption that the coefficient is Lipschitz-continuous, further improving the applicability of the regularity theory. Alberti [17] obtained Hölder regularity results with a small Hölder exponent $\alpha \in (0, 1/2]$ under minimal smoothness assumptions on the coefficients, and we extend the Hölder regularity to the case of an arbitrary Hölder exponent $\alpha \in (0, 1)$ that is consistent with the regularity of the coefficient matrices, which is a significant improvement of the existing results.

Corollary 4.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded, connected and simply connected open set with a $C^{1,1}$ -smooth connected boundary $\partial\Omega$, and $A(x), B(x) \in C^\alpha(\overline{\Omega})$ with $\alpha = \frac{\tau-1}{2}$, $B(x) \in C^0(\Omega)$ and $f, h, \nabla \times g \in L^{2,\tau}(\Omega)$ for $\tau \in (1, 3)$. Let $u \in L^2(\Omega)$ be a weak solution to system (45), then

$$\| \nabla \times u \|_{C^\alpha(\overline{\Omega})} \leq C \left(\| (u, f, h, \nabla \times g) \|_{L^{2,t}_\lambda(\Omega)} + \| G \|_{C^\alpha(\overline{\Omega})} \right) \tag{46}$$

Corollary 4.1 applies our main theorems of the anisotropic div-curl system to the time-harmonic Maxwell system (43), and it establishes the Hölder regularity of the electric and magnetic fields under the condition that the permittivity $\varepsilon(x)$ and permeability $\mu(x)$ are Hölder-continuous. It shows that the electromagnetic field inherits the Hölder regularity from the permittivity and permeability coefficients of the anisotropic medium, which is significant in the study of electromagnetic wave propagation in anisotropic media and the optimal design of electromagnetic devices. This result not only relaxes the smoothness requirements of the medium coefficients from Lipschitz continuity to Hölder continuity, but also removes the additional structural constraints on the coefficient matrices in existing studies, making the regularity theory of the Maxwell system more in line with the characteristics of practical anisotropic electromagnetic media.

Proof. Let $v = \nabla \times u - G$, then v satisfies

$$\{\nabla \times (A(x)v) + B(x)u = f + \nabla \times g - \nabla \times (A(x)G) \quad \text{in } \Omega$$
$$\nabla \cdot v = h - \nabla \cdot G \quad \text{in } \Omega \tag{47}$$
$$\mathbf{n} \times v = 0 \quad \text{on } \partial\Omega$$

We apply Theorem 2.6 for system (47),

$$\| v \|_{C^{\alpha}(\overline{\Omega})} \leq C\left(\| (u, f, h, \nabla \times g) \|_{L^{2,t}_{\lambda}(\Omega)} + \| G \|_{C^{\alpha}(\overline{\Omega})} \right) \tag{48}$$

and $\nabla \times u = v + G$

$$\| \nabla \times u \|_{C^{\alpha}(\overline{\Omega})} = \| v + G \|_{C^{\alpha}(\overline{\Omega})} \leq C\left(\| (u, f, h, \nabla \times g) \|_{L^{2,t}_{\lambda}(\Omega)} + \| G \|_{C^{\alpha}(\overline{\Omega})} \right) \tag{49}$$

Similarly, when $\varepsilon \in C^{\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, the magnetic field H in the Maxwells system (43) has the estimate as follows:

$$\| \nabla \times H \|_{C^{\alpha}(\overline{\Omega})} \leq C\left(\| (H, J_m) \|_{L^{2,t}_{\lambda}(\Omega)} + \| (G, J_e) \|_{C^{\alpha}(\overline{\Omega})} \right) \tag{50}$$

CONCLUSIONS

This work resolves a long-standing open problem by establishing that weak solutions to the Div-Curl system exhibit Hölder continuity when the coefficients are anisotropic and possess limited regularity. This accomplishment directly addresses the inquiry posited by Yin. The research extends pertinent conclusions. The findings of this investigation not only resolve the issue in the context of a single coefficient but also successfully generalize to the scenario involving two anisotropic coefficients, confirming that the Hölder regularity of the solution persists in this context. This investigation establishes a precise correlation, explicitly demonstrating a positive relationship between the regularity of solutions and that of coefficients.

One of the key observations in this paper is that the regularity of solutions is fully determined by the regularity of coefficients, which confirms a typical regularity inheritance mechanism in elliptic and div-curl type systems. When the coefficients enjoy higher-order Hölder continuity, the solutions automatically gain the same improvement in smoothness, which is consistent with the classical regularity theory of linear partial differential equations.

Furthermore, the method developed here can be extended to more general settings, such as systems in higher-dimensional spaces, systems with discontinuous coefficients, and systems with mixed boundary conditions. These potential extensions will be the focus of our future research.

The fundamental conclusion is: should the coefficients exhibit Hölder continuity, then Hölder-continuous coefficients imply Hölder-continuous solutions. Should the coefficient possess Hölder regularity of a higher order, the solution's Hölder regularity will correspondingly increase, indicating that the regularity of the solution will be inherited and enhanced as the regularity of the coefficient improves.

Author Contributions

Conceptualization – Yikun Bai, and Yu Lei; methodology – Yu Lei; investigation – Yu Lei; resources – Yikun Bai; writing-original draft preparation – Yu Lei; writing-review and editing – Yikun Bai; visualization – Yikun Bai, and Yu Lei; supervision – Basang Tering-xiao. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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